

ON THE CALABI-YAU EQUATION IN THE KODAIRA-THURSTON MANIFOLD

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ABSTRACT. We review some previous results about the Calabi-Yau equation on the Kodaira-Thurston manifold equipped with an invariant almost-Kähler structure and assuming the volume form T^2 -invariant. In particular, we observe that under some restrictions the problem is reduced to a Monge-Ampère equation by using the ansatz $\tilde{\omega} = \Omega - dJdu + da$, where u is a T^2 -invariant function and a is a 1-form depending on u . Furthermore, we extend our analysis to non-invariant almost-complex structures by considering some basic cases and we finally take into account a generalization to higher dimensions.

1. INTRODUCTION

The *Calabi-Yau problem* in 4-dimensional almost-Kähler manifolds is a PDEs system arising from the generalization of the classical Calabi-Yau theorem to the non-Kähler setting.

The Calabi-Yau theorem [14] states that on a compact Kähler manifold (X, J, Ω) for every smooth function $F: X \rightarrow \mathbb{R}$ such that

$$(1) \quad \int_X e^F \Omega^n = \int_X \Omega^n$$

there always exists a unique Kähler form $\tilde{\omega}$ on (X, J) satisfying

$$(2) \quad [\tilde{\omega}] = [\Omega], \quad \tilde{\omega}^n = e^F \Omega^n.$$

An analogue problem still makes sense in the almost-Kähler case, when J is merely an almost-complex structure and Ω is a J -compatible symplectic form. It turns out that in this more general context, the PDEs system arising from (2) is overdetermined for $n \geq 3$, while it is elliptic in dimension 4 (see [3]). Consequently, the Calabi-Yau problem is mainly studied in 4-dimensional almost-Kähler manifolds (see [1, 2, 10, 11, 12, 15] and the references therein).

The study of the problem is strongly motivated by a project of Donaldson involving compact symplectic 4-manifolds (see [3]). The project is based on a conjecture stated in [3] and partially confirmed by Taubes in [13].

In [15] Weinkove attacked the problem by introducing a *symplectic potential*. Indeed, given two almost-Kähler forms Ω and $\tilde{\omega}$ on a compact almost-complex manifold (X, J)

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satisfying $[\Omega] = [\tilde{\omega}]$ there always exists a function u , called the *symplectic potential*, such that

$$(\tilde{\omega} - \Omega) \wedge \tilde{\omega} = -dJdu \wedge \tilde{\omega}.$$

In terms of u one can always write

$$\tilde{\omega} = \Omega - dJdu + da,$$

where a is a 1-form which can be assumed co-closed with respect to the co-differential induced by $\tilde{\omega}$ (in this way a is unique up addition of *harmonic* 1-forms).

Weinkove proved that in order to show the solvability of the Calabi-Yau problem (2) it's enough to provide an a priori estimate on the C^0 -norm of the almost-Kähler potential (see theorem 1 in [15]); that can be always done if the L^1 -norm of the Nijenhuis tensor of J is small enough (see theorem 2 in [15]).

In [12] Tosatti and Weinkove studied the Calabi-Yau problem on the Kodaira-Thurston manifold (M, Ω_0, J_0) showing that under the assumption on the initial datum F to be invariant by the action of a 2-dimensional torus the problem has a unique solution. The Kodaira-Thurston manifold M is a 4-dimensional 2-step nilmanifold carrying a natural almost-Kähler structure and it can be viewed as a torus bundle over a torus (more precisely M is an S^1 -bundle over a 3-dimensional torus).

In [4] it is observed that if F is T^2 -invariant, then (2) on the Kodaira-Thurston manifold M can be rewritten in terms of the Monge-Ampère equation

$$(1 + u_{xx})(1 + u_{yy}) - u_{xy}^2 = e^F$$

on the 2-dimensional torus \mathbb{T}_{xy}^2 and the Tosatti-Weinkove result in [12] can be alternatively obtained by applying a result of Y.Y. Li in [8]. A similar approach was then adopted in [1, 4] in order to study the Calabi-Yau problem in every 4-dimensional torus bundle over a torus equipped with an invariant almost-Kähler structure. In this more general case the equation writes in terms of a “modified” Monge-Ampère equation which is still solvable. Furthermore, in [2] it is studied the equation on the Kodaira-Thurston manifold when F is S^1 -invariant (instead of T^2 -invariant as in the previous papers). It turns out that in this last case the Calabi-Yau problem writes as a PDE on the 3-dimensional torus \mathbb{T}_{xyt}^3 which is not of Monge-Ampère type anymore.

In this paper we review some results in [4] showing that when the projection is Lagrangian, the reduction of the Calabi-Yau problem on the Kodaira-Thurston manifold to a scalar PDE can be obtained by setting

$$\tilde{\omega} = \Omega + d(-Ju + u\gamma_1 + u_y\gamma_2)$$

where γ_1 and γ_2 are suitable invariant forms depending on (Ω, J) , u is in the same space of F and y is a coordinate on the base.

In section 3 we study the Calabi-Yau equation on (M, Ω_0) for S^1 -invariant almost complex structures J compatible to Ω_0 . Under some strong restrictions on J , the equation can be still reduced to a PDE in a single unknown function. In section 4 we prove the solvability of the arising equations in some special cases leaving the more general cases for an eventually future work.

In the last section we consider a generalization of the previous sections to 2-step nilmanifold in higher dimensions.

A remark on the notation. If P is an m -torus bundle over an n -torus, we denote by \mathbb{T}^n the base of P and by T^m the principal fiber, in order to distinguish the base and the fibers.

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2. CALABI-YAU EQUATIONS ON THE KODAIRA-THURSTON MANIFOLD

In this section we review some results in [1, 2, 4] about the Calabi-Yau equation on the Kodaira-Thurston manifold. The *Kodaira-Thurston manifold* is a compact 2-step nilmanifold M defined as the quotient $M = \Gamma \backslash G$, where G is the Lie group given by \mathbb{R}^4 in the variables (x_1, x_2, y_1, y_2) with the multiplication

$$(x_1, x_2, y_1, y_2) \cdot (x'_1, x'_2, y'_1, y'_2) = (x_1 + x'_1, x_2 + x'_2, y_1 + y'_1, y_2 + y'_2 + x_1 x'_2)$$

and Γ is the co-compact lattice given by \mathbb{Z}^4 with the induced multiplication. Alternatively M can be defined as the product $M = \Gamma_0 \backslash \text{Nil}^3 \times S^1$, where Nil^3 is the 3-dimensional real Heisenberg group

$$\text{Nil}^3 = \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \mid x, y, z \in \mathbb{R} \right\}$$

and Γ_0 is the lattice in Nil^3 of matrices having integers entries. M has a natural structure of principal S^1 -bundle over a 3-dimensional torus \mathbb{T}^3 induced by the map $[x_1, x_2, y_1, y_2] \mapsto [x_1, x_2, y_1]$ and it is parallelizable. A global co-frame on M is for instance given by

$$e^1 = dx_1, \quad e^2 = dx_2, \quad f^1 = dy_1, \quad f^2 = dy_2 - x_1 dx_2.$$

For such co-frame we have

$$de^1 = de^2 = df^1 = 0, \quad df^2 = -e^1 \wedge e^2$$

and its dual basis is given by $\{\partial_{x_1}, \partial_{x_2} + x_1 \partial_{y_2}, \partial_{y_1}, -\partial_{y_2}\}$. Furthermore, M has the “natural” almost-Kähler structure (Ω_0, J_0) given by the symplectic form

$$(3) \quad \Omega_0 = e^1 \wedge f^1 + e^2 \wedge f^2$$

and the Riemannian metric

$$(4) \quad g_0 = e^1 \otimes e^1 + f^1 \otimes f^1 + e^2 \otimes e^2 + f^2 \otimes f^2.$$

The following proposition is proved in [2]

Proposition 2.1. *Let $u: M \rightarrow \mathbb{R}$ be an S^1 -invariant function and*

$$\alpha := -J_0 du - ue^1.$$

Then

$$d\alpha \text{ is of type } (1, 1)$$

and

$$(5) \quad (\Omega_0 + d\alpha)^2 = (\det(I + \mathcal{A}(u)) - u_{x_2 y_1}^2) \Omega_0^2,$$

where I is the identity 2×2 matrix and

$$(6) \quad \mathcal{A}(u) = \begin{pmatrix} u_{x_1 x_1} + u_{y_1 y_1} + u_{y_1} & u_{x_1 x_2} \\ u_{x_1 x_2} & u_{x_2 x_2} \end{pmatrix}.$$

Proof. Let $u: M \rightarrow \mathbb{R}$ be an S^1 -invariant function. Then

$$du = u_{x_1} e^1 + u_{x_2} e^2 + u_{y_1} f^1$$

and

$$-J_0 du = u_{x_1} f^1 + u_{x_2} f^2 - u_{y_1} e^1$$

and

$$-dJ_0 du = \sum_{i,j=1}^2 u_{x_i x_j} e^i \wedge f^j + u_{x_2 y_1} e^1 \wedge e^2 + u_{x_2 y_1} f^1 \wedge f^2 + u_{y_1 y_1} e^1 \wedge f^1 - u_{x_2} e^1 \wedge e^2.$$

Therefore, if $\alpha = -J_0 du - u e^1$, we have

$$\begin{aligned} d\alpha &= -dJ_0 du - du \wedge e^1 \\ &= \sum_{i,j=1}^2 u_{x_i x_j} e^i \wedge f^j + u_{x_2 y_1} e^1 \wedge e^2 + u_{x_2 y_1} f^1 \wedge f^2 + u_{y_1 y_1} e^1 \wedge f^1 + u_{y_1} e^1 \wedge f^1 \end{aligned}$$

which is a form of type $(1, 1)$ with respect to J_0 . Formula (5) follows from a straightforward computation. \square

Proposition 2.1 is useful in the study of the Calabi-Yau problem on (M, Ω_0, J_0) . Indeed, let $F: M \rightarrow \mathbb{R}$ be an S^1 -invariant function satisfying $\int_M e^F \Omega_0^2 = 1$ and consider the Calabi-Yau equation $(\Omega_0 + d\alpha)^2 = e^F \Omega_0^2$ on (M, Ω_0, J_0) . In view of proposition 2.1, we can study the Calabi-Yau problem by introducing the ansatz

$$\alpha = -J_0 du - u e^1$$

where u is an unknown S^1 -invariant map. In this way the Calabi-Yau problem reduces to the single equation

$$(7) \quad \det(I + \mathcal{A}(u)) - u_{x_2 y_1}^2 = e^F,$$

on the 3-dimensional torus $\mathbb{T}_{x_1 x_2 y_1}^3$, where $\mathcal{A}(u)$ is given by (6). The main result in [2] is the following

Theorem 2.2. *Equation (7) has a solution for every S^1 -invariant initial datum $F: M \rightarrow \mathbb{R}$ satisfying $\int_M e^F \Omega_0^2 = 1$. Consequently the Calabi-Yau problem $(\Omega_0 + d\alpha)^2 = e^F \Omega_0^2$ has a unique solution for every S^1 -invariant function $F: M \rightarrow \mathbb{R}$.*

Special cases of equation (7) occur when we see M as a 2-torus bundle over a 2-dimensional torus and we assume F depending only on the coordinates of the base. Those cases correspond to assume F depending either on (x_1, x_2) or on (x_2, y_1) (the case $F = F(x_1, y_1)$ is equivalent to $F = F(x_2, y_1)$).

If $F = F(x_1, x_2)$, we can assume u depending only on (x_1, x_2) and (7) reduces to the Monge-Ampère type equation

$$(8) \quad (1 + u_{x_1 x_1})(1 + u_{x_2 x_2}) - u_{x_1 x_2}^2 = e^F$$

on the 2-dimensional torus $\mathbb{T}_{x_1x_2}^2$. This equation has a solution in view of a theorem of Y.Y. Li (see [8]). Note that in this case the solution u to (8) is an almost-Kähler potential of $\tilde{\omega} = \Omega_0 + d\alpha$ with respect to Ω_0 . Indeed,

$$\tilde{\omega} = (1 + u_{x_1x_1})e^1 \wedge f^1 + (1 + u_{x_2x_2})e^2 \wedge f^2 + u_{x_1x_2}e^1 \wedge f^2 + u_{x_1x_2}f^1 \wedge e^2$$

and

$$\tilde{\omega} - \Omega_0 = -dJ_0du + da$$

where

$$a = -ue^1.$$

Hence $da = u_{x_2}e^1 \wedge e^2$ and

$$\tilde{\omega} \wedge da = 0$$

which implies

$$(\tilde{\omega} - \Omega_0) \wedge \tilde{\omega} = -dJ_0du \wedge \tilde{\omega}.$$

If $F = F(x_2, y_1)$, we assume u depending only on (x_2, y_1) and (7) reduces to the “modified” Monge-Ampère equation

$$(9) \quad (1 + u_{y_1y_1} + u_{y_1})(1 + u_{x_2x_2}) - u_{x_2y_1}^2 = e^F$$

on the 2-dimensional torus $\mathbb{T}_{x_2y_1}^2$. The existence of a solution to this last equation was proved in [4]. Note that in this case

$$\tilde{\omega} = (1 + u_{y_1y_1} + u_{y_1})e^1 \wedge f^1 + (1 + u_{x_2x_2})e^2 \wedge f^2 + u_{x_2y_1}e^1 \wedge e^2 + u_{x_2y_1}f^1 \wedge f^2$$

and if u solves (9), then

$$d\alpha = -dJ_0du + da,$$

where $da = -u_{x_2}e^1 \wedge e^2 - u_{y_1}e^1 \wedge f^1$. Therefore

$$da \wedge \tilde{\omega} \neq 0$$

and u is not an almost-Kähler potential.

Next, we take into account the Calabi-Yau problem on M viewed as a 2-torus bundle over a 2-torus equipped with an invariant Lagrangian almost-Kähler structure (Ω, J) and we assume F defined on the base. Here by *Lagrangian* we mean that the fibers of the fibration are Lagrangian submanifolds.

Proposition 2.3. *Let (Ω, J) be an invariant almost-Kähler structure on M . Then there exist real numbers μ_1 and μ_2 and an invariant 1-form β such that if $u = u(x_1, x_2)$ is a smooth function on M , then*

$$\alpha = -Jdu + \mu_1ue^1 - \mu_2ue^2 - u_y\beta$$

is such that $d\alpha$ is of type $(1, 1)$. Moreover

$$(10) \quad (\Omega + d\alpha)^2 = \frac{1}{l_1l_2} ((l_1 + u_{x_1x_1})(l_2 + u_{x_2x_2}) - (u_{x_1x_2})^2) \Omega^2.$$

where l_1 and l_2 are positive real numbers.

Proof. We set $x_1 = x$ and $x_2 = y$ in order to simplify the notation. We can find an invariant Hermitian coframe $\{\alpha^1, \alpha^2, \beta^1, \beta^2\}$ on M such that

$$\Omega = \alpha^1 \wedge \beta^1 + \alpha^2 \wedge \beta^2$$

and

$$dx = A\alpha^1, \quad dy = B\alpha^1 + C\alpha^2.$$

Note that $dx \wedge dy = AC\alpha^1 \wedge \alpha^2$ and we can write

$$d\beta^1 = \lambda_1 dx \wedge dy, \quad d\beta^2 = \lambda_2 dx \wedge dy$$

for some λ_1, λ_2 in \mathbb{R} . Now

$$du = u_x dx + u_y dy = (Au_x + Bu_y)\alpha^1 + Cu_y\alpha^2$$

and

$$-Jdu = (Au_x + Bu_y)\beta^1 + Cu_y\beta^2$$

So

$$-dJdu = Au_{xx}dx \wedge \beta^1 + Au_{xy}dy \wedge \beta^1 + Cu_{xy}dx \wedge \beta^2 + Cu_{yy}dy \wedge \beta^2 + d(\gamma + Bu_y\beta^1)$$

where

$$\gamma = \lambda_1 Au dy - \lambda_2 Cu dx.$$

Hence

$$\begin{aligned} -dJdu &= A^2 u_{xx} \alpha^1 \wedge \beta^1 + AB u_{xy} \alpha^1 \wedge \beta^1 + AC u_{xy} \alpha^2 \wedge \beta^1 + AC u_{xy} \alpha^1 \wedge \beta^2 \\ &\quad + BC u_{yy} \alpha^1 \wedge \beta^2 + C^2 u_{yy} \alpha^2 \wedge \beta^2 + d(Bu_y \alpha^2 + \gamma). \end{aligned}$$

which implies that

$$\alpha = -Jdu - Bu_y \alpha^2 - \gamma$$

is such that $d\alpha$ is of type $(1, 1)$.

Moreover,

$$\begin{aligned} (\Omega + d\alpha)^2 &= ((1 + A^2 u_{xx})(1 + C^2 u_{yy}) - (AC u_{xy})^2) \Omega^2 \\ &= \frac{1}{l_1 l_2} ((l_1 + u_{xx})(l_2 + u_{yy}) - (u_{xy})^2) \Omega^2 \end{aligned}$$

where $l_1 = 1/A^2$ and $l_2 = 1/C^2$ and the claim follows. \square

Proposition 2.4. *Let (Ω, J) be an invariant almost-Kähler structure on M which is Lagrangian with respect to the fibration $[x_1, x_2, y_1, y_2] \mapsto [x_2, y_1]$. There exist invariant 1-forms γ^1, γ^2 such that if $u = u(x_2, y_1)$ is a smooth function on M , then*

$$\alpha = -Jdu + u\gamma^1 + u_{y_1}\gamma^2$$

is such that $d\alpha$ is of type $(1, 1)$. Moreover

$$(11) \quad (\Omega + d\alpha)^2 = \frac{1}{l_1 l_2} ((l_1 + u_{x_2 x_2})(l_2 + u_{y_1 y_1} + m_1 u_{x_2} + m_2 u_{y_1}) - (u_{x_2 y_1})^2) \Omega^2$$

where $l_1, l_2, m_1, m_2 \in \mathbb{R}$ and $l_1, l_2 < 0$.

Proof. First of all we use that (Ω, J) is an invariant almost-Kähler structure on M which is Lagrangian with respect to $[x_1, x_2, y_1, y_2] \mapsto [x_2, y_1]$, then there exists an invariant Hermitian co-frame $\{\alpha^1, \alpha^2, \beta^1, \beta^2\}$ on M such that

$$\Omega = \alpha^1 \wedge \beta^1 + \alpha^2 \wedge \beta^2$$

and

$$\alpha^2 \in \langle e^2 \rangle, \quad \beta^1 \in \langle e^2, f^1 \rangle, \quad \alpha^1 \in \langle e^1, e^2, f^1 \rangle$$

(see lemma 5.1 in [4]). In this way

$$dx_2 = A\alpha^2, \quad dy_1 = B\alpha^2 + C\beta^1, \quad d\beta^2 = \lambda\alpha^1 \wedge \beta^1 + \mu\alpha^2 \wedge \beta^1$$

for some $A, B, C, \lambda, \mu \in \mathbb{R}$. In order to simplify the notation we set $x_2 = x$ and $y_1 = y$. Then

$$du = u_x dx + u_y dy = Au_x \alpha^2 + u_y (B\alpha^2 + C\beta^1) = (Au_x + Bu_y)\alpha^2 + Cu_y \beta^1$$

and

$$-Jdu = -(Au_x + Bu_y)\beta^2 + Cu_y \alpha^1.$$

So

$$\begin{aligned} -dJdu &= -Au_{xx} dx \wedge \beta^2 - Au_{xy} dy \wedge \beta^2 + Cu_{xy} dx \wedge \alpha^1 \\ &\quad + Cu_{yy} dy \wedge \alpha^1 - (Au_x + Bu_y) (\lambda \alpha^1 \wedge \beta^1 + \mu \alpha^2 \wedge \beta^1) - d(Bu_y \beta^2) \end{aligned}$$

i.e.,

$$\begin{aligned} -dJdu &= -A^2 u_{xx} \alpha^2 \wedge \beta^2 - BAu_{xy} \alpha^2 \wedge \beta^2 - ACu_{xy} \beta^1 \wedge \beta^2 + ACu_{xy} \alpha^2 \wedge \alpha^1 \\ &\quad + CBu_{yy} \alpha^2 \wedge \alpha^1 + C^2 u_{yy} \beta^1 \wedge \alpha^1 - (Au_x + Bu_y) (\lambda \alpha^1 \wedge \beta^1 + \mu \alpha^2 \wedge \beta^1) - d(Bu_y \beta^2). \end{aligned}$$

Now,

$$(Au_x + Bu_y) (\lambda \alpha^1 \wedge \beta^1 + \mu \alpha^2 \wedge \beta^1) = \lambda (Au_x + Bu_y) \alpha^1 \wedge \beta^1 + d(\mu u \beta^1)$$

and we can write

$$\begin{aligned} -dJdu &= (-C^2 u_{yy} - \lambda Au_x - \lambda Bu_y) \alpha^1 \wedge \beta^1 + (-A^2 u_{xx} - BAu_{xy}) \alpha^2 \wedge \beta^2 \\ &\quad - ACu_{xy} \beta^1 \wedge \beta^2 - (ACu_{xy} + B^2 u_{yy}) \alpha^1 \wedge \alpha^2 - d(\mu u \beta^1 + Bu_y \beta^2) \end{aligned}$$

which implies the first part of the statement.

Moreover,

$$\begin{aligned} (\Omega + d\alpha)^2 &= ((1 - A^2 u_{xx})(1 - C^2 u_{yy} - \lambda Au_x - \lambda Bu_y) - (ACu_{xy})^2) \Omega^2 \\ &= \frac{1}{l_1 l_2} ((l_1 + u_{xx})(l_2 + u_{yy} + m_1 u_x + m_2 u_y) - (u_{xy})^2) \Omega^2 \end{aligned}$$

where

$$l_1 = -\frac{1}{A^2}, \quad l_2 = -\frac{1}{C^2}, \quad m_1 = -\lambda \frac{A}{C^2}, \quad m_2 = -\lambda \frac{B}{C^2}$$

and the claim follows. \square

From propositions 2.3 and 2.4 it follows that if we see M as 2-torus over a 2-torus and we fix an invariant Lagrangian almost-Kähler structure (Ω, J) on M ; then for every given F defined on the base of M and satisfying $\int_M e^F \Omega^2 = \int_M \Omega^2$ the corresponding Calabi-Yau equation can be written in terms of an unknown function u on the base \mathbb{T}_{xy}^2 of M as

$$\frac{1}{l_1 l_2} ((l_1 + u_{xx})(l_2 + u_{yy} + m_1 u_x + m_2 u_y) - (u_{xy})^2) = e^F$$

where $l_1, l_2, m_1, m_2 \in \mathbb{R}$ and l_1 and l_2 are both positive or negative. This kind of equations are solvable in view of theorem 6.2 in [4].

3. THE EQUATION FOR NON-INVARIANT ALMOST-COMPLEX STRUCTURES

As pointed out in [12] it is interesting to extend the results described in the previous section to torus-invariant almost complex structures on the Kodaira-Thurston manifold M which are compatible to “natural” symplectic form Ω_0 defined in (3). In this section we consider some basic cases. Let $h = h(x_1, y_1)$ be a function in $C^\infty(\mathbb{T}_{x_1 y_1}^2)$ and consider the family of Ω_0 -compatible almost-complex structures J_h induced by the relations

$$(12) \quad J_h(e^1) = -e^h f^1 \quad J_h(e^2) = -f^2.$$

The following result is a generalization of proposition 2.1 to the family J_h .

Proposition 3.1. *Let $u: M \rightarrow \mathbb{R}$ be an S^1 -invariant function and*

$$\alpha := -J_h du - u e^1.$$

Then

$$d\alpha \text{ is of type } (1, 1)$$

and

$$(13) \quad (\Omega_0 + d\alpha)^2 = (\det(I + \mathcal{A}_h(u)) - e^{-h} u_{x_2 y_1}^2) \Omega_0^2$$

where I is the identity 2×2 matrix and

$$(14) \quad \mathcal{A}_h(u) = \begin{pmatrix} e^h u_{x_1 x_1} + e^{-h} u_{y_1 y_1} + u_{y_1} + e^h h_{x_1} u_{x_1} - e^{-h} h_{y_1} u_{y_1} & u_{x_1 x_2} \\ e^h u_{x_1 x_2} & u_{x_2 x_2} \end{pmatrix}$$

Proof. Let u be an S^1 -invariant function. Then

$$-J_h du = e^h u_{x_1} f^1 + u_{x_2} f^2 - e^{-h} u_{y_1} e^1$$

and

$$\begin{aligned} -dJ_h du &= (e^h u_{x_1})_{x_1} e^1 \wedge f^1 + e^h u_{x_1 x_2} e^2 \wedge f^1 + u_{x_1 x_2} e^1 \wedge f^2 + u_{x_2 x_2} e^2 \wedge f^2 \\ &\quad + u_{x_2 y_1} f^1 \wedge f^2 + e^{-h} u_{x_2 y_1} e^1 \wedge e^2 + (e^{-h} u_{y_1})_{y_1} e^1 \wedge f^1 - u_{x_2} e^1 \wedge e^2, \end{aligned}$$

i.e.,

$$\begin{aligned} -dJ_h du &= (e^h u_{x_1 x_1} + e^{-h} u_{y_1 y_1} + e^h h_{x_1} u_{x_1} - e^{-h} h_{y_1} u_{y_1} + u_{y_1}) e^1 \wedge f^1 + u_{x_2 x_2} e^2 \wedge f^2 \\ &\quad + e^h u_{x_1 x_2} e^2 \wedge f^1 + u_{x_1 x_2} e^1 \wedge f^2 + u_{x_2 y_1} f^1 \wedge f^2 + (e^{-h} u_{x_2 y_1} - u_{x_2}) e^1 \wedge e^2. \end{aligned}$$

Therefore if $\alpha = -J_h du - ue^1$, then

$$\begin{aligned} d\alpha = & (e^h u_{x_1 x_1} + e^{-h} u_{y_1 y_1} + e^h h_{x_1} u_{x_1} - e^{-h} h_{y_1} u_{y_1} + u_{y_1}) e^1 \wedge f^1 + u_{x_2 x_2} e^2 \wedge f^2 \\ & + e^h u_{x_1 x_2} e^2 \wedge f^1 + u_{x_1 x_2} e^1 \wedge f^2 + u_{x_2 y_1} f^1 \wedge f^2 + e^{-h} u_{x_2 y_1} e^1 \wedge e^2. \end{aligned}$$

which is of type $(1, 1)$ and

$$(\Omega_0 + d\alpha)^2 = \det(I + \mathcal{A}_h(u)) - e^{-h} u_{x_2 y_1}^2,$$

as required. \square

In view of proposition 3.1, the Calabi-Yau equation on (M, Ω_0, J_h) , for an S^1 -invariant function $F: M \rightarrow \mathbb{R}$ can be reduced to

$$(15) \quad \det(I + \mathcal{A}_h(u)) - e^{-h} u_{x_2 y_1}^2 = e^F$$

where \mathcal{A}_h is given by (14) and $u: M \rightarrow \mathbb{R}$ is an unknown S^1 -invariant function. Note that for $h = 0$, equation (15) reduces to equation (7) studied in [2]. We consider the following special cases:

If $h = h(x_1)$ and $F = F(x_1, x_2)$ we may assume u depending only on (x_1, x_2) and (15) reduces in the variables $x = x_1, y = x_2$ to

$$\det \begin{pmatrix} 1 + e^h u_{xx} + e^h h' u_x & u_{xy} \\ e^h u_{xy} & 1 + u_{yy} \end{pmatrix} = e^F$$

on the 2-dimensional torus \mathbb{T}_{xy}^2 . Such an equation can be rewritten as

$$\det \begin{pmatrix} e^{-h} + u_{xx} + h' u_x & u_{xy} \\ u_{xy} & 1 + u_{yy} \end{pmatrix} = e^{F-h}.$$

If $h = h(y_1)$ and $F = F(x_2, y_1)$, then we assume u depending only on (x_2, y_1) and (15) reduces in the variables $x = y_1, y = x_2$ to

$$\det \begin{pmatrix} 1 + e^{-h} u_{xx} + (1 - e^{-h} h') u_x & u_{xy} \\ e^{-h} u_{xy} & 1 + u_{yy} \end{pmatrix} = e^F$$

on \mathbb{T}_{xy}^2 . Such an equation can be rewritten as

$$\det \begin{pmatrix} e^h + u_{xx} + (e^h - h') u_x & u_{xy} \\ u_{xy} & 1 + u_{yy} \end{pmatrix} = e^{F+h}.$$

Both cases fit in the following class of equations on \mathbb{T}_{xy}^2

$$\det \begin{pmatrix} e^{-h} + u_{xx} + (ce^{-h} + h') u_x & u_{xy} \\ u_{xy} & 1 + u_{yy} \end{pmatrix} = e^{F-h}$$

where $h = h(x)$ is a smooth 1-periodic functions on \mathbb{R} and $c \in \mathbb{R}$. We will show the solvability of the last class of equations in the next section.

4. SOLVABILITY OF THE SPECIAL CASES

The aim of this section is to prove the following result

Theorem 4.1. *Let $h = h(x)$ be a smooth 1-periodic functions on \mathbb{R} , $c \in \mathbb{R}$ and let $F = F(x, y) \in C^\infty(\mathbb{T}^2)$ be such that*

$$\int_{\mathbb{T}^2} e^F dx \wedge dy = 1.$$

Then equation

$$(16) \quad \det \begin{pmatrix} e^{-h} + u_{xx} + (ce^{-h} + h')u_x & u_{xy} \\ u_{xy} & 1 + u_{yy} \end{pmatrix} = e^{F-h}$$

has a solution $u \in C^\infty(\mathbb{T}^2)$.

Before proving theorem 4.1 we consider the following preliminary lemma which is a slight generalization of lemma 6.3 in [4].

Lemma 4.2. *Let $h, v \in C^1(\mathbb{R})$ be 1-periodic functions satisfying*

$$e^h v' + (c + e^h h')v > -1.$$

Assume there exists $s_0 \in [0, 1]$ such that $v(s_0) = 0$; then

$$\|v\|_{C^0} \leq C,$$

where C is a constant depending only on c and h .

Proof. Let G be a primitive of $ce^{-h} + h'$ in \mathbb{R} . Since

$$v' + (ce^{-h} + h')v > -e^{-h},$$

in terms of G we have

$$e^G(v' + G'v) > -e^{G-h},$$

i.e.

$$\frac{d}{ds}(e^G v) > -e^{G-h}.$$

Since $v(s_0) = 0$, we have

$$\int_{s_0}^s \frac{d}{ds}(e^G v) ds > - \int_{s_0}^s e^{G-h} d\tau, \quad \text{for every } s \geq 1,$$

which implies

$$v(s) > -e^{-G(s)} \int_{s_0}^s e^{G-h} d\tau, \quad \text{for every } s \in [1, 2].$$

On the other hand

$$\int_s^{s_0} \frac{d}{ds}(e^G v) ds > - \int_s^{s_0} e^{G-h} d\tau, \quad \text{for every } s \leq 0,$$

which implies

$$v(s) < e^{-G(s)} \int_s^{s_0} e^{G-h} d\tau, \quad \text{for every } s \in [-1, 0].$$

The claim follows since v is 1-periodic. □

Now we can prove theorem 4.1

Proof of Theorem 4.1. Fix $0 < \alpha < 1$ and let $C_0^{2,\alpha}(\mathbb{T}^2)$ be the space of $C^{2,\alpha}$ -functions u on \mathbb{T}^2 satisfying

$$\int_{\mathbb{T}^2} u \, dx \wedge dy = 0.$$

Then we consider the operator $T: C_0^{2,\alpha}(\mathbb{T}^2) \times [0, 1] \rightarrow C_0^{0,\alpha}(\mathbb{T}^2)$ defined by

$$T(u, t) = \det \begin{pmatrix} e^{-h} + u_{xx} + (ce^{-h} + h')u_x & u_{xy} \\ u_{xy} & 1 + u_{yy} \end{pmatrix} - e^{-h} (te^F + 1 - t)$$

in order that $u \in C_0^{2,\alpha}(\mathbb{T}^2)$ solves (16) if and only if $T(u, 1) = 0$. Then we define the set

$$S := \{t \in [0, 1] : \text{there exists } u \in C_0^{2,\alpha}(\mathbb{T}^2) \text{ such that } T(u, t) = 0\}.$$

Note that S is not empty since $u \equiv 0$ satisfies $T(u, 0) = 0$. We will show that $1 \in S$ by proving that S is open and closed in $[0, 1]$. In this way we get that (16) has a solution u in $C^{2,\alpha}(\mathbb{T}^2)$ and theorem 3 in [9] implies that u is in fact C^∞ . Note that if $(u, t) \in C_0^2(\mathbb{T}^2) \times [0, 1]$ is such that $T(u, t) = 0$, then the matrix

$$\mathcal{A}_h := \det \begin{pmatrix} e^{-h} + u_{xx} + (ce^{-h} + h')u_x & u_{xy} \\ u_{xy} & 1 + u_{yy} \end{pmatrix}$$

is positive-defined. Indeed, since $\int_{\mathbb{T}^2} e^F dx \wedge dy = 0$, then $\mathcal{A}_h(u)$ is non-singular and at a minimum point of u all the eigenvalues of \mathcal{A}_h are positive.

Now we prove that S is closed. First of all we observe that if $u \in C_0^2(\mathbb{T}^2)$ satisfies $T(u, t) = 0$ for some $t \in [0, 1]$, then

$$(17) \quad e^h u_{xx} + (c + e^h h')u_x > -1,$$

$$(18) \quad 1 + u_{yy} > -1.$$

Indeed, since

$$(1 + e^h u_{xx} + (c + e^h h')u_x)(1 + u_{yy}) > 0$$

the two terms have the same sign, and they are both positive at a point (x_0, y_0) where u reaches its minimum value. Lemma 4.2 then implies

$$(19) \quad \|u_x\|_{C^0} \leq C \quad \text{and} \quad \|u_y\|_{C^0} \leq C$$

where C is a constant depending on c, h and k . Now we focus on the C^0 estimate on u . Let (x_0, y_0) be a point in $[0, 1] \times [0, 1]$ where u vanishes, then

$$\begin{aligned} u(x, y) &= (x - x_0) \int_0^1 u_x((1 - t)x + tx_0, (1 - t)y + ty_0) \, dt + \\ &\quad (y - y_0) \int_0^1 u_y((1 - t)x + tx_0, (1 - t)y + ty_0) \, dt, \end{aligned}$$

and by using (19) we get

$$|u(x, y)| \leq C(x - x_0) + C(y - y_0)$$

which readily implies

$$\|u\|_{C^0} \leq C.$$

Hence u satisfies a C^1 a priori bound. Furthermore, if $t \in [0, 1]$ is fixed, equation

$$T(u, t) = 0$$

belongs to the class of equations studied in [7] and theorem 2 in [7] implies that if $u \in C_0^{2,\alpha}(\mathbb{T}^2)$ solves $T(u, t) = 0$ for some t and satisfies a priori C^1 bound, then it also satisfies a $C^{2,\alpha}$ bound. This implies that S is closed in $[0, 1]$. Indeed, let t_n be a sequence in S converging to \bar{t} in $[0, 1]$. To each t_n corresponds a function $u_n \in C_0^{2,\alpha}(\mathbb{T}^2)$ such that $T(u_n, t_n) = 0$. The $C^{2,\alpha}$ a priori bound on solutions to $T(u, t) = 0$ implies that the sequence u_n is bounded in $C_0^{2,\alpha}(\mathbb{T}^2)$ and so it admits a subsequence, which we still denote by u_n , which converges in $C_0^2(\mathbb{T}^2)$ to a function $\bar{u} \in C_0^2(\mathbb{T}^2)$. Since T is continuous, $T(\bar{u}, \bar{t}) = 0$ and so, in view of [7], \bar{u} in $C^{2,\alpha}(\mathbb{T}^2)$. Hence $\bar{t} \in S$ and S is closed.

Next we show that S is open. Let $t_0 \in S$. Then there exists $u \in C_0^{2,\alpha}(\mathbb{T}^2)$ such that $T(u, t_0) = 0$. Let $L: C_0^{2,\alpha}(\mathbb{T}^2) \rightarrow C_0^{0,\alpha}(\mathbb{T}^2)$ be defined as

$$L(w) := T_{*|(u, t_0)}(w, 0).$$

A direct computation yields that

$$(20) \quad L(w) = (w_{xx} + (ce^{-h} + h')w_x)(1 + u_{yy}) \\ + (e^{-h} + u_{xx} + (ce^{-h} + h')u_x)(w_{yy}) - 2u_{xy}w_{xy}$$

and so L is uniformly elliptic. L is injective by maximum principle and it is surjective in view of elliptic theory (see e.g. [5]). Therefore the implicit function theorem implies that \bar{t} has a open neighborhood contained in S , and so S is open, as required. \square

5. A GENERALIZATION TO HIGHER DIMENSIONS

In this section we consider a generalization of the Kodaira-Thurston manifold in dimension greater than 4. Assume $n \geq 3$. Let G_n be the Lie group $(\mathbb{R}^{2n}, *_n)$, where

$$(x_1, \dots, x_n, y_1, \dots, y_n) *_n (x'_1, \dots, x'_n, y'_1, \dots, y'_n) = \\ (x_1 + x'_1, \dots, x_n + x'_n, y_1 + y'_1, y_2 + y'_2 - x_2 x'_1, \dots, y_{n-1} + y'_{n-1} - x_n x'_1)$$

and let $M_n = \Gamma_n \backslash G_n$, where Γ_n is \mathbb{Z}^{2n} with the multiplication induced by $*_n$. Then M_n is a 2-step nilmanifold and the projection $\pi: \mathbb{R}^{2n} \rightarrow \mathbb{R}^{n+1}$ onto the first $(n+1)$ -coordinates induces to M_n a structure of principal $(n-1)$ -torus bundle on an $(n+1)$ -torus \mathbb{T}^{n+1} . M_n is parallelizable and

$$e^i = dx_i, \quad i = 1, \dots, n, \quad f^j = dy_j - x_1 dx_j, \quad j = 1, \dots, n$$

defines a global coframe which satisfies

$$de^k = 0, \quad k = 1, \dots, n, \quad df^1 = 0, \quad df^k = e^k \wedge e^1, \quad k = 2, \dots, n.$$

We then consider on M_n the symplectic form

$$\Omega_n = \sum_{k=1}^n \alpha^k \wedge \beta^k$$

and the Ω_n -compatible almost-complex structure J_n induced by Ω_n and the natural metric

$$g_n = \sum_{k=1}^n \alpha^k \otimes \alpha^k + \beta^k \otimes \beta^k.$$

In terms of the basis $\mathcal{B} = \{e^1, \dots, e^n, f^1, \dots, f^n\}$, J_n is defined by

$$J_n e^k = -f^k, \quad J_n f^k = e^k.$$

Let u be a T^{n+1} -invariant function on M_n ; then

$$du = \sum_{s=1}^n u_{x_s} e^s + u_{y_1} f^1, \quad -J_n du = \sum_{s=1}^n u_{x_s} f^s - u_{y_1} e^1$$

and so

$$\begin{aligned} -dJ_n du &= \sum_{r,s=1}^n u_{x_r x_s} e^r \wedge f^s - \sum_{k=1}^n u_{x_k y_1} e^k \wedge e^1 + u_{x_k y_1} f^1 \wedge f^k + u_{y_1 y_1} e^1 \wedge f^1 + \sum_{k=2}^n u_{x_r} e^r \wedge e^1 \\ &= \sum_{r,s=1}^n u_{x_r x_s} e^r \wedge f^s - \sum_{k=1}^n u_{x_k y_1} e^k \wedge e^1 + u_{x_k y_1} f^1 \wedge f^k + (u_{y_1 y_1} - u_{y_1}) e^1 \wedge f^1 + d(ue^1), \end{aligned}$$

and so if

$$\alpha = -J_n du - ue^1,$$

then $d\alpha$ is of type $(1, 1)$ with respect to J_n . Furthermore,

$$\begin{aligned} (\Omega_n + d\alpha)^n &= \left(\sum_{r,s=1}^n (\delta_{rs} + u_{x_r x_s}) e^r \wedge f^s + (1 + u_{y_1 y_1} - u_{y_1}) e^1 \wedge f^1 \right)^n \\ &\quad - n! \sum_{k,m=2}^n \left(\prod_{r,s=2, (r,s) \neq (k,m)}^n u_{x_r x_s} u_{x_k y_1} u_{x_m y_1} \right) e^1 \wedge f^1 \wedge \dots \wedge e^n \wedge f^n \end{aligned}$$

and if F is a given T^{n+1} -invariant map, the Calabi-Yau equation $(\Omega + d\alpha)^n = e^F \Omega^n$ reads in terms of u as

$$(21) \quad \det(I + \mathcal{A}(u)) - \sum_{k,m=2}^n \left(\prod_{r,s=2, (r,s) \neq (k,m)}^n u_{x_r x_s} u_{x_k y_1} u_{x_m y_1} \right) = e^F,$$

where $\mathcal{A}(u) = (A_{ij})$ is the $n \times n$ matrix

$$A_{11} = u_{x_1 x_1} + u_{y_1 y_1} - u_{y_1}, \quad A_{ij} = u_{x_i x_j}, \quad \text{if } (i, j) \neq (1, 1).$$

Example 5.1. For $n = 3$, equation (21) reads as

$$\det(I + A(u)) - u_{x_3 x_3} u_{x_2 y_1}^2 - u_{x_2 x_2} u_{x_3 y_1}^2 - 2u_{x_2 x_3} u_{x_2 y_1} u_{x_3 y_1} = e^F,$$

this kind of equations has been considered in [6].

In analogy to the case $n = 2$, we can obtain special cases by regarding M_n as a principal T^n -bundle over a \mathbb{T}^n and assuming F to be T^n -invariant. It is not restrictive considering only the following two cases:

$$F = F(x_1, \dots, x_n), \quad \text{or } F = F(x_2, \dots, x_n, y_1).$$

- In the first case $F = F(x_1, \dots, x_n)$, equation (21) reduces to the Monge-Ampère equation

$$\det(I + \mathcal{H}(u)) = e^F$$

on the n -dimensional torus $\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$, where $\mathcal{H}(u)$ is the Hessian metric of u . In this case the equation has a solution in view of [8].

- In the second case, $F = F(x_2, \dots, x_n, y_1)$, in the variables

$$z_1 = y_1, z_2 = x_2, \dots, z_n = x_n$$

equation (21) take the following expression

$$\det(I + \mathcal{B}(u)) = e^F$$

where $\mathcal{B}(u) = (B_{ij})$ is given by

$$B_{11} = u_{z_1 z_1} + u_{z_1}, \quad B_{ij} = u_{z_i z_j}, \quad \text{if } i, j \neq 1.$$

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